J. Math. Anal. Appl. 363 (2010) 559–568

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Necessary conditions for L^1-convergence of double Fourier series

Ferenc Móricz

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged 6720, Hungary

ARTICLE INFO

Article history:
Received 21 January 2009
Available online 22 September 2009
Submitted by L. Grafakos

Keywords:
Double Fourier series
L^1-convergence
Hardy's inequality for functions in the Hardy space H^1
Bernstein–Zygmund inequalities for the derivatives of trigonometric polynomials in L^1-norm
Conjugate trigonometric polynomials

ABSTRACT

We extend the results of A.S. Belov from single to double Fourier series, which give necessary conditions in terms of the Fourier coefficients for L^1-convergence. Our basic tools are Hardy's inequality for the Taylor coefficients of a function in the Hardy space H^1 on the unit disk, and the Bernstein–Zygmund inequalities for the derivative of a trigonometric polynomial in L^1-norm.

1. Known results on single Fourier series

Let $f = f(x)$ be a complex-valued function, periodic with period 2π, and integrable in Lebesgue's sense, in symbols: $f \in L^1(\mathbb{T})$. We consider its Fourier series

$$f(x) \in \sum_{k\in\mathbb{Z}} c_k e^{ikx}, \quad x \in \mathbb{T} := [-\pi, \pi), \tag{1.1}$$

where the Fourier coefficients c_k are defined by

$$c_k := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx}\, dx, \quad k \in \mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

The (symmetric) partial sums $s_m(f)$ of the series in (1.1) are defined by

$$s_m(f) = s_m(f; x) := \sum_{|k|\leqslant m} c_k e^{ikx}, \quad m \in \mathbb{N} := \{0, 1, 2, \ldots\}.$$

As usual, the $L^1(\mathbb{T})$-norm of f is defined by

$$\|f\| = \|f(x)\|_1 := \frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|\, dx.$$

E-mail address: moricz@math.u-szeged.hu.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.09.030

Belov [1,2] proved necessary conditions in terms of the Fourier coefficients for the $L^1(\mathbb{T})$-convergence (or $L^1(\mathbb{T})$-boundedness) of the partial sums, in symbols:

$$\left\| s_m(f) - f \right\| = o(1) \quad \bigl(\text{or } O(1)\bigr). \tag{1.2}$$

Theorem 1. *(See [1, Theorem 1].) Suppose $f \in L^1(\mathbb{T})$ and (1.2) holds. Then we have*

$$\sum_{k=[m/2]}^{2m} \frac{|c_k| + |c_{-k}|}{|k-m|+1} = o(1) \quad \bigl(\text{or } O(1)\bigr), \tag{1.3}$$

where $[\cdot]$ means the integer part of a real number.

The following corollary of Theorem 1 says that the L^1-convergence of a Fourier series implies that the average of the absolute values of the Fourier coefficients c_k for $m \leqslant |k| \leqslant 2m$ decays faster that $\ln m$ as $m \to \infty$.

Corollary 1. *(See [2, p. 816].) Suppose (1.3) holds. Then we have*

$$\frac{\ln m}{m} \sum_{k=m}^{2m} \bigl(|c_k| + |c_{-k}|\bigr) = o(1) \quad \bigl(\text{or } O(1)\bigr), \tag{1.4}$$

where ln *is the natural logarithm.*

Clearly, (1.4) is equivalent to the following statement:

$$\frac{1}{m+1} \sum_{k=m}^{2m} \bigl(|c_k| + |c_{-k}|\bigr) \ln k = o(1) \quad \bigl(\text{or } O(1)\bigr). \tag{1.5}$$

Observe that the left-hand side is the moving average of the sequence $\{(|c_k| + |c_{-k}|) \ln k\colon\ k = 1, 2, \ldots\}$. Clearly, (1.5) is satisfied if

$$\bigl(|c_k| + |c_{-k}|\bigr) \ln k = o(1) \quad \bigl(\text{or } O(1)\bigr),$$

but the converse statement fails in general.

2. New results on double Fourier series

Our goal is to extend Theorem 1 and Corollary 1 from single to double Fourier series.

Let $f = f(x, y)$ be a complex-valued function, periodic with period 2π in each variable, and integrable in Lebesgue's sense, in symbols: $f \in L^1(\mathbb{T}^2)$. We consider its double Fourier series

$$f(x, y) \sim \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} c_{k\ell} e^{i(kx + \ell y)}, \quad (x, y) \in \mathbb{T}^2, \tag{2.1}$$

where the double Fourier coefficients are defined by

$$c_{k\ell} := \frac{1}{4\pi^2} \int\!\!\int_{\mathbb{T}^2} f(x, y) e^{-i(kx + \ell y)}\, dx\, dy, \quad (k, \ell) \in \mathbb{Z}^2.$$

The (symmetric) rectangular partial sums of the double series in (2.1) are defined by

$$s_{mn}(f) = s_{mn}(f; x, y) := \sum_{|k| \leqslant m} \sum_{|\ell| \leqslant n} c_{k\ell} e^{i(kx + \ell y)}, \quad (m, n) \in \mathbb{N}^2.$$

As usual, the $L^1(\mathbb{T}^2)$-norm of f is defined by

$$\|f\| = \bigl\| f(x, y) \bigr\|_1 := \frac{1}{4\pi^2} \int\!\!\int_{\mathbb{T}^2} \bigl| f(x, y) \bigr|\, dx\, dy.$$

The extensions of Theorem 1 and Corollary 1 from single to double Fourier series read as follows.

Theorem 2. *Suppose $f \in L^1(\mathbb{T}^2)$ and*

$$\|s_{mn}(f) - f\| \to 0 \quad \text{as } m, n \to \infty \tag{2.2}$$

independently of one another. Then we have

$$\sum_{k=[m/2]}^{2m} \sum_{\ell=[n/2]}^{2n} \frac{|c_{k\ell}| + |c_{-k,\ell}| + |c_{k,-\ell}| + |c_{-k,-\ell}|}{(|k-m|+1)(|\ell-n|+1)} \to 0 \quad \text{as } m, n \to \infty. \tag{2.3}$$

The following corollary of Theorem 2 says that the L^1-convergence of a double Fourier series implies that the average of the absolute values of the Fourier coefficients $c_{k\ell}$ when $(|k|, |\ell|)$ belongs to the rectangle with vertices (m, n), $(2m, n)$, $(2m, 2n)$, $(m, 2n)$ decays faster than $(\ln m)(\ln n)$ as $m, n \to \infty$.

Corollary 2. *Suppose* (2.3) *holds. Then we have*

$$\frac{(\ln m)(\ln n)}{mn} \sum_{k=m}^{2m} \sum_{\ell=n}^{2n} \big(|c_{k\ell}| + |c_{-k,\ell}| + |c_{k,-\ell}| + |c_{-k,-\ell}|\big) \to 0 \quad \text{as } m, n \to \infty. \tag{2.4}$$

It is obvious that (2.4) is equivalent to the following statement:

$$\frac{1}{(m+1)(n+1)} \sum_{k=m}^{2m} \sum_{\ell=n}^{2n} \big(|c_{k\ell}| + |c_{-k,\ell}| + |c_{k,-\ell}| + |c_{-k,-\ell}|\big)(\ln k)(\ln \ell) \to 0 \quad \text{as } m, n \to \infty. \tag{2.5}$$

Observe that the left-hand side is the moving average over rectangles. It is easy to see that (2.5) is satisfied if

$$\big(|c_{k\ell}| + |c_{-k,\ell}| + |c_{k,-\ell}| + |c_{-k,-\ell}|\big)(\ln k)(\ln \ell) \to 0 \quad \text{as } k, \ell \to \infty,$$

but the converse statement fails in general.

Remark 1. Condition (2.3) is only necessary, but not sufficient for the $L^1(\mathbb{T}^2)$-convergence of the Fourier series in question. To justify this claim, consider the cosine series

$$\sum_{k=3}^{\infty} \frac{\cos kx}{\sqrt{\ln k}} = \sum_{k=3}^{\infty} \frac{e^{ikx} + e^{-ikx}}{2\sqrt{\ln k}} =: f_1(x). \tag{2.6}$$

By Kolmogorov's theorem (see, e.g., [4, Chapter V, Theorems 1.5 and 1.12 on pp. 183–185]), $f_1 \in L^1(\mathbb{T})$, the series in (2.6) is the Fourier series of its sum f_1, and

$$\left\| \sum_{k=3}^{m} \frac{\cos kx}{\sqrt{\ln k}} - f_1(x) \right\| \to \infty \quad \text{as } m \to \infty.$$

Now, let

$$c_{k\ell} := \begin{cases} \frac{1}{2\sqrt{\ln |k|}} & \text{for } |k| \geqslant 3 \text{ and } \ell = 0, \\ 0 & \text{otherwise, where } (k, \ell) \in \mathbb{Z}^2; \end{cases}$$

and let f be the sum of the double series in (2.1) with these coefficients $c_{k\ell}$. Clearly, $f \in L^1(\mathbb{T}^2)$ and for each $n \geqslant 0$,

$$\|s_{mn}(f) - f\| = \|s_m(f_1) - f_1\| \to \infty \quad \text{as } m \to \infty;$$

while condition (2.3) is trivially satisfied.

Remark 2. It follows from (2.2) that

$$\sup_{m,n \geqslant M} \|s_{mn}(f) - f\| \to 0 \quad \text{as } M \to \infty.$$

Consequently, for large enough M depending on f, we have

$$\sup_{m,n \geqslant M} \|s_{mn}(f)\| < \infty.$$

In connection with this, we raise the following

Problem. Find a function $f \in L^1(\mathbb{T}^2)$ such that condition (2.2) is satisfied and

$$\sup_{(m,n)\in\mathbb{N}^2} \left\| s_{mn}(f) \right\| = \infty.$$

We note that in the case of double series $\sum\sum c_{k\ell}$ of complex numbers boundedness of the rectangular partial sums does not follow from their convergence. For example, let

$$c_{k\ell} := \begin{cases} k & \text{if } k \in \mathbb{N} \text{ and } \ell = 0, \\ -k & \text{if } k \in \mathbb{N} \text{ and } \ell = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, for $m \in \mathbb{N}$,

$$s_{mn} := \sum_{|k|\leqslant m} \sum_{|\ell|\leqslant n} c_{k\ell} = \begin{cases} \frac{m(m+1)}{2} & \text{if } n = 0, \\ 0 & \text{if } n \geqslant 1; \end{cases}$$

and consequently, we have

$$\sup_{(m,n)\in\mathbb{N}^2} |s_{mn}| = \infty \quad \text{and} \quad s_{mn} \to 0 \quad \text{as } m, n \to \infty.$$

Remark 3. The L^1-convergence of the double Fourier series in (2.1) which is defined in (2.2) may be called rectangular L^1-convergence. In the special case when $m = n$, the $s_{mm}(f) = s_{mm}(f; x, y)$ are called the (symmetric) square partial sums of the double Fourier series in (2.1). Accordingly, if $m = n$ in (2.2), the L^1-convergence of the double Fourier series in (2.1) may be called square L^1-convergence as $m = n \to \infty$. However, Theorem 2 is no longer true in this case.

It would be also natural to sum a double Fourier series using circular regions instead of rectangular ones. We recall that the circular partial sums of the Fourier series in (2.1) are defined by

$$s_r(f) = s_r(f; x, y) := \sum_{k^2+\ell^2\leqslant r^2} c_{k\ell} e^{i(kx+\ell y)}, \quad r = 0, 1, 2, \ldots.$$

Accordingly, one may consider the circular L^1-convergence of the double Fourier series in (2.1) as $r \to \infty$. As is well known, the circular and rectangular partial sums of a double Fourier series behave quite differently in many respects. Thus, we will not deal with the square or circular L^1-convergence in this paper.

3. Auxiliary results

One of our main tools in proving Theorem 2 is Hardy's inequality, according to which if a power series belongs to the Hardy space H^1 on the open unit disk:

$$\varphi(z) := \sum_{k=0}^{\infty} c_k z^k \in H^1(\mathbb{D}), \quad z \in \mathbb{D} := \left\{ z \in \mathbb{C}\colon |z| < 1 \right\},$$

then

$$\left\| \sum_{k=0}^{\infty} c_k e^{ikx} \right\| \geqslant \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{|c_k|}{k+1}$$

(see, e.g., [3, p. 48] or [4, Chapter VII, Theorem 8.7]). In the sequel, $\varphi(z)$ will always be a polynomial of z, which clearly belongs to H^1.

Lemma 1. *For every $m \in \mathbb{N}$, we have*

$$\min\left\{ \left\| \sum_{k=0}^{m} c_k e^{ikx} \right\|, \left\| \sum_{k=0}^{m} c_k e^{-ikx} \right\| \right\} \geqslant \frac{1}{\pi} \max\left\{ \sum_{k=0}^{m} \frac{|c_k|}{k+1}, \sum_{k=0}^{m} \frac{|c_k|}{m-k+1} \right\}.$$

Proof. This lemma is an immediate consequence of Hardy's inequality. Indeed, we have

$$2\pi\left\|\sum_{k=0}^{m}c_k e^{ikx}\right\| := \int_{\mathbb{T}}\left|\sum_{k=0}^{m}c_k e^{ikx}\right|dx = \int_{\mathbb{T}}\left|\sum_{k=0}^{m}\bar{c}_k e^{-ikx}\right|dx = \int_{\mathbb{T}}\left|e^{imx}\sum_{k=0}^{m}\bar{c}_k e^{-ikx}\right|dx$$
$$= \int_{\mathbb{T}}\left|\sum_{k=0}^{m}\bar{c}_k e^{i(m-k)x}\right|dx \geqslant 2\sum_{k=0}^{m}\frac{|c_k|}{m-k+1}.$$

The inequalities for $\sum c_k e^{ix}$ are proved.

The inequalities for $\sum c_k e^{-ikx}$ can be proved analogously, since

$$\left\|\sum_{k=0}^{m}c_k e^{-ikx}\right\| = \left\|\sum_{k=0}^{m}\bar{c}_k e^{ikx}\right\|. \qquad \square$$

Lemma 2. *For all $(m, n) \in \mathbb{N}^2$, we have*

$$\min\left\{\left\|\sum_{k=0}^{m}\sum_{\ell=0}^{n}c_{k\ell}e^{i(kx+\ell y)}\right\|, \left\|\sum_{k=0}^{m}\sum_{\ell=0}^{n}c_{k\ell}e^{i(-kx+\ell y)}\right\|, \left\|\sum_{k=0}^{m}\sum_{\ell=0}^{n}c_{k\ell}e^{i(kx-\ell y)}\right\|, \left\|\sum_{k=0}^{m}\sum_{\ell=0}^{n}c_{k\ell}e^{i(-kx-\ell y)}\right\|\right\}$$
$$\geqslant \frac{1}{\pi^2}\max\left\{\sum_{k=0}^{m}\sum_{\ell=0}^{n}\frac{|c_{k\ell}|}{(k+1)(\ell+1)}, \sum_{k=0}^{m}\sum_{\ell=0}^{n}\frac{|c_{k\ell}|}{(m-k+1)(\ell+1)},\right.$$
$$\left.\sum_{k=0}^{m}\sum_{\ell=0}^{n}\frac{|c_{k\ell}|}{(k+1)(n-\ell+1)}, \sum_{k=0}^{m}\sum_{\ell=0}^{n}\frac{|c_{k\ell}|}{(m-k+1)(n-\ell+1)}\right\}.$$

Proof. Applying Lemma 1 twice together with Fubini's theorem on successive integration yields, for example, the following inequality:

$$4\pi^2\left\|\sum_{k=0}^{m}\sum_{\ell=0}^{n}c_{k\ell}e^{i(kx+\ell y)}\right\| := \iint_{\mathbb{T}^2}\left|\sum_{k=0}^{m}\sum_{\ell=0}^{n}c_{k\ell}e^{i(kx+\ell y)}\right|dx\,dy$$
$$= \int_{\mathbb{T}}\left(\int_{\mathbb{T}}\left|\sum_{k=0}^{m}\left(\sum_{\ell=0}^{n}c_{k\ell}e^{i\ell y}\right)e^{ikx}\right|dx\right)dy$$
$$\geqslant 2\int_{\mathbb{T}}\sum_{k=0}^{m}\frac{1}{k+1}\left|\sum_{\ell=0}^{n}c_{k\ell}e^{i\ell y}\right|dy$$
$$= 2\sum_{k=0}^{m}\frac{1}{k+1}\int_{\mathbb{T}}\left|\sum_{\ell=0}^{n}c_{k\ell}e^{i\ell y}\right|dy$$
$$\geqslant 4\sum_{k=0}^{m}\frac{1}{k+1}\sum_{\ell=0}^{n}\frac{|c_{k\ell}|}{\ell+1}.$$

The proofs of the other fifteen inequalities run along the same lines. $\square$

Our second main tool in proving Theorem 2 is the Bernstein–Zygmund inequalities, which we formulate as follows:

$$\max\left\{\left\|\frac{d}{dx}\sum_{|k|\leqslant m}c_k e^{ikx}\right\|, \left\|\frac{d}{dx}\sum_{|k|\leqslant m}(-i\,\mathrm{sign}\,k)c_k e^{ikx}\right\|\right\} \leqslant m\left\|\sum_{|k|\leqslant m}c_k e^{ikx}\right\|$$

(see, e.g., [4, Chapter X, formula (3.18) on p. 11 and the particular case of formula (3.25) on p. 13]). We note that the trigonometric polynomial

$$\sum_{|k|\leqslant m}(-i\,\mathrm{sign}\,k)c_k e^{ikx} \text{ is said to be the conjugate one to } \sum_{|k|\leqslant m}c_k e^{ikx}.$$

Lemma 3. *For all $(m,n) \in \mathbb{N}^2$, we have*

$$\max\Bigg\{\Bigg\|\frac{\partial^2}{\partial x\partial y}\sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n} c_{k\ell}e^{i(kx+\ell y)}\Bigg\|, \Bigg\|\frac{\partial^2}{\partial x\partial y}\sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n}(-i\operatorname{sign}k)c_{k\ell}e^{i(kx+\ell y)}\Bigg\|,$$
$$\Bigg\|\frac{\partial^2}{\partial x\partial y}\sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n}(-i\operatorname{sign}\ell)c_{k\ell}e^{i(kx+\ell y)}\Bigg\|, \Bigg\|\frac{\partial^2}{\partial x\partial y}\sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n}(-i\operatorname{sign}k)(-i\operatorname{sign}\ell)c_{k\ell}e^{i(kx+\ell y)}\Bigg\|\Bigg\}$$
$$\leqslant mn\Bigg\|\sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n} c_{k\ell}e^{i(kx+\ell y)}\Bigg\|.$$

Proof. Repeated applications of Fubini's theorem and the first Bernstein–Zygmund inequality yield, for example, the following inequality:

$$\begin{aligned}
4\pi^2\Bigg\|\frac{\partial^2}{\partial x\partial y}\sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n} c_{k\ell}e^{i(kx+\ell y)}\Bigg\| &= \int_{\mathbb{T}}\Bigg(\int_{\mathbb{T}}\Bigg|\frac{\partial}{\partial x}\sum_{|k|\leqslant m}\Bigg(\frac{\partial}{\partial y}\sum_{|\ell|\leqslant n} c_{k\ell}e^{i\ell y}\Bigg)e^{ikx}\Bigg|\,dx\Bigg)\,dy \\
&\leqslant \int_{\mathbb{T}}\Bigg(m\int_{\mathbb{T}}\Bigg|\sum_{|k|\leqslant m}\Bigg(\frac{\partial}{\partial y}\sum_{|\ell|\leqslant n} c_{k\ell}e^{i\ell y}\Bigg)e^{ikx}\Bigg|\,dx\Bigg)\,dy \\
&= m\int_{\mathbb{T}}\Bigg(\int_{\mathbb{T}}\Bigg|\frac{\partial}{\partial y}\sum_{|\ell|\leqslant n}\Bigg(\sum_{|k|\leqslant m} c_{k\ell}e^{ikx}\Bigg)e^{i\ell y}\Bigg|\,dy\Bigg)\,dx \\
&\leqslant m\int_{\mathbb{T}}\Bigg(n\int_{\mathbb{T}}\Bigg|\sum_{|\ell|\leqslant n}\Bigg(\sum_{|k|\leqslant m} c_{k\ell}e^{ikx}\Bigg)e^{i\ell y}\Bigg|\,dy\Bigg)\,dx \\
&= mn\iint_{\mathbb{T}^2}\Bigg|\sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n} c_{k\ell}e^{i(kx+\ell y)}\Bigg|\,dx\,dy \\
&=: 4\pi^2\Bigg\|\sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n} c_{k\ell}e^{i(kx+\ell y)}\Bigg\|.
\end{aligned}$$

The other three inequalities can be proved analogously. □

Lemma 4. *For all $-1 \leqslant m < \mu$ and $-1 \leqslant n < \nu$, we have*

$$\max\Bigg\{\Bigg\|\sum_{k=m+1}^{\mu}\sum_{\ell=n+1}^{\nu} k\ell c_{k\ell}e^{i(kx+\ell y)}\Bigg\|, \Bigg\|\sum_{k=m+1}^{\mu}\sum_{\ell=n+1}^{\nu}(-k)\ell c_{-k,\ell}e^{i(-kx+\ell y)}\Bigg\|,$$
$$\Bigg\|\sum_{k=m+1}^{\mu}\sum_{\ell=n+1}^{\nu} k(-\ell)c_{k,-\ell}e^{i(kx-\ell y)}\Bigg\|, \Bigg\|\sum_{k=m+1}^{\mu}\sum_{\ell=n+1}^{\nu}(-k)(-\ell)c_{-k,-\ell}e^{i(-kx-\ell y)}\Bigg\|\Bigg\}$$
$$\leqslant \mu\nu\|s_{\mu\nu} - s_{m\nu} - s_{\mu n} + s_{mn}\|,$$

where $s_{-1,n} = s_{m,-1} = s_{-1,-1} := 0$ and

$$s_{mn} := \sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n} c_{k\ell}e^{i(kx+\ell y)}, \quad (m,n) \in \mathbb{N}^2.$$

Proof. We introduce the following notations:

$$\tilde{s}_{mn}^{(1,0)} := \sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n}(-i\operatorname{sign}k)c_{k\ell}e^{i(kx+\ell y)},$$
$$\tilde{s}_{mn}^{(0,1)} := \sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n}(-i\operatorname{sign}\ell)c_{k\ell}e^{i(kx+\ell y)},$$
$$\tilde{s}_{mn}^{(1,1)} := \sum_{|k|\leqslant m}\sum_{|\ell|\leqslant n}(-i\operatorname{sign}k)(-i\operatorname{sign}\ell)c_{k\ell}e^{i(kx+\ell y)};$$

furthermore, we set

$$s(m,n;\mu,\nu) := s_{\mu\nu} - s_{m\nu} - s_{\mu n} + s_{mn} = \sum_{m<|k|\leqslant\mu} \sum_{n<|\ell|\leqslant\nu} c_{k\ell} e^{i(kx+\ell y)},$$

$$\tilde{s}^{(1,0)}(m,n;\mu,\nu) := \tilde{s}^{(1,0)}_{\mu\nu} - \tilde{s}^{(1,0)}_{m\nu} - \tilde{s}^{(1,0)}_{\mu n} + \tilde{s}^{(1,0)}_{mn},$$

$\tilde{s}^{(0,1)}(m,n;\mu,\nu)$ and $\tilde{s}^{(1,1)}(m,n;\mu,\nu)$ are defined analogously.

It is easy to see that

$$\begin{aligned}
&\frac{\partial^2}{\partial x\partial y}\big\{s(m,n;\mu,\nu) + i\tilde{s}^{(1,0)}(m,n;\mu,\nu) + i\tilde{s}^{(0,1)}(m,n;\mu,\nu) + i^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\big\} \\
&= \sum_{m<|k|\leqslant\mu} \sum_{n<|\ell|\leqslant\nu} k\ell c_{k\ell} e^{i(kx+\ell y)} \big\{i^2 + i^3(-i\operatorname{sign} k) + i^3(-i\operatorname{sign}\ell) + i^4(-i\operatorname{sign} k)(-i\operatorname{sign}\ell)\big\} \\
&= \sum_{m<|k|\leqslant\mu} \sum_{n<|\ell|\leqslant\nu} k\ell c_{k\ell} e^{i(kx+\ell y)} \big\{-1 - i\operatorname{sign} k - i\operatorname{sign}\ell - (\operatorname{sign} k)(\operatorname{sign}\ell)\big\} \\
&= -4 \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} k\ell c_{k\ell} e^{i(kx+\ell y)}.
\end{aligned}$$

In an analogous way, we obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial x\partial y}\big\{s(m,n;\mu,\nu) - i\tilde{s}^{(1,0)}(m,n;\mu,\nu) + i\tilde{s}^{(0,1)}(m,n;\mu,\nu) - i^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\big\} \\
&= \sum_{m<|k|\leqslant\mu} \sum_{n<|\ell|\leqslant\nu} k\ell c_{k\ell} e^{i(kx+\ell y)} \big\{-1 + \operatorname{sign} k - \operatorname{sign}\ell + (\operatorname{sign} k)(\operatorname{sign}\ell)\big\} \\
&= -4 \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} (-k)\ell c_{-k,\ell} e^{i(-kx+\ell y)},
\end{aligned}$$

its symmetric counterpart

$$\begin{aligned}
&\frac{\partial^2}{\partial x\partial y}\big\{s(m,n;\mu,\nu) + i\tilde{s}^{(1,0)}(m,n;\mu,\nu) - i\tilde{s}^{(0,1)}(m,n;\mu,\nu) - i^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\big\} \\
&= -4 \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} k(-\ell) c_{k,-\ell} e^{i(kx-\ell y)},
\end{aligned}$$

and finally, we obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial x\partial y}\big\{s(m,n;\mu,\nu) - i\tilde{s}^{(1,0)}(m,n;\mu,\nu) - i\tilde{s}^{(0,1)}(m,n;\mu,\nu) + i^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\big\} \\
&= \sum_{m<|k|\leqslant\mu} \sum_{n<|\ell|\leqslant\nu} k\ell e^{i(kx+\ell y)} \big\{-1 + \operatorname{sign} k + \operatorname{sign}\ell - (\operatorname{sign} k)(\operatorname{sign}\ell)\big\} \\
&= -4 \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} (-k)(-\ell) c_{-k,-\ell} e^{i(-kx-\ell y)}.
\end{aligned}$$

Next, we apply Lemma 3 four times. For example, we find that

$$\begin{aligned}
&4\Bigg\| \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} k\ell c_{k\ell} e^{i(kx+\ell y)} \Bigg\| \\
&= \bigg\| \frac{\partial^2}{\partial x\partial y}\big\{s(m,n;\mu,\nu) + i\tilde{s}^{(1,0)}(m,n;\mu,\nu) + i\tilde{s}^{(0,1)}(m,n;\mu,\nu) + i^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\big\} \bigg\| \\
&\leqslant \bigg\| \frac{\partial^2}{\partial x\partial y} s(m,n;\mu,\nu) \bigg\| + \bigg\| \frac{\partial^2}{\partial x\partial y} \tilde{s}^{(1,0)}(m,n;\mu,\nu) \bigg\| + \bigg\| \frac{\partial^2}{\partial x\partial y} \tilde{s}^{(0,1)}(m,n;\mu,\nu) \bigg\| + \bigg\| \frac{\partial^2}{\partial x\partial y} \tilde{s}^{(1,1)}(m,n;\mu,\nu) \bigg\| \\
&\leqslant 4\mu\nu \big\| s(m,n;\mu,\nu) \big\| = 4\mu\nu \| s_{\mu\nu} - s_{m\nu} - s_{\mu n} + s_{mn} \|.
\end{aligned}$$

The other three inequalities in Lemma 4 are similarly obtained. $\square$

Lemma 5. *For $0 \leqslant m < \mu$ and $0 \leqslant n < \nu$, we have*

$$\left\| \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} k\ell c_{k\ell} e^{i(kx+\ell y)} \right\| \geqslant \frac{1}{\pi^2} \max \left\{ \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} \frac{k\ell |c_{k\ell}|}{(k-m)(\ell-n)}, \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} \frac{k\ell |c_{k\ell}|}{(\mu-k+1)(\ell-n)}, \right.$$
$$\left. \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} \frac{k\ell |c_{k\ell}|}{(k-m)(\nu-\ell+1)}, \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} \frac{k\ell |c_{k\ell}|}{(\mu-k+1)(\nu-\ell+1)} \right\},$$

and three other analogous inequalities involving $|c_{-k,\ell}|$, $|c_{k,-\ell}|$ and $|c_{-k,-\ell}|$, respectively, in place of $|c_{k\ell}|$.

Proof. We may equally write that

$$\begin{aligned}
\pi^2 \left\| \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} k\ell c_{k\ell} e^{i(kx+\ell y)} \right\| &= \pi^2 \left\| e^{i((m+1)x+(n+1)y)} \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} k\ell c_{k\ell} e^{i((k-m-1)x+(\ell-n+1)y)} \right\| \\
&= \pi^2 \left\| e^{i(\mu x+(n+1)y)} \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} k\ell c_{k\ell} e^{i((k-\mu)x+(\ell-n+1)y)} \right\| \\
&= \pi^2 \left\| e^{i((m+1)x+\nu y)} \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} k\ell c_{k\ell} e^{i((k-m-1)x+(\ell-\nu)y)} \right\| \\
&= \pi^2 \left\| e^{i(\mu x+\nu y)} \sum_{k=m+1}^{\mu} \sum_{\ell=n+1}^{\nu} k\ell c_{k\ell} e^{i((k-\mu)x+(\ell-\nu)y)} \right\| \\
&= \pi^2 \left\| \sum_{k_1=0}^{\mu-m-1} \sum_{\ell_1=0}^{\nu-n-1} k\ell c_{k\ell} e^{i(k_1 x+\ell_1 y)} \right\| \\
&= \pi^2 \left\| \sum_{k_1=0}^{\mu-m-1} \sum_{\ell_1=0}^{\nu-n-1} k\ell c_{k\ell} e^{i(-(\mu-k_1-m-1)x+\ell_1 y)} \right\| \\
&= \pi^2 \left\| \sum_{k_1=0}^{\mu-m-1} \sum_{\ell_1=0}^{\nu-n-1} k\ell c_{k\ell} e^{i((k-\mu)x-(\nu-\ell_1-n-1)y)} \right\| \\
&= \pi^2 \left\| \sum_{k_1=0}^{\mu-m-1} \sum_{\ell_1=0}^{\nu-n-1} k\ell c_{k\ell} e^{i(-(\mu-k_1-m-1)x-(\nu-\ell_1-n-1)y)} \right\|,
\end{aligned}$$

where $k_1 := k-m-1$ and $\ell_1 := \ell-n-1$. Applying Lemma 4 yields the first inequality in Lemma 5 (indicated in details).

The other three analogous inequalities involving $|c_{-k,\ell}|$, $|c_{k,-\ell}|$ and $|c_{-k,-\ell}|$, respectively, in place of $|c_{k\ell}|$ are proved by the same reasoning. □

4. Proofs of the main results

Proof of Theorem 2. For the sake of brevity in writing, set

$$C_{k\ell} := |c_{k\ell}| + |c_{-k,\ell}| + |c_{k,-\ell}| + |c_{-k,-\ell}|, \quad (k,\ell) \in \mathbb{N}^2. \tag{4.1}$$

Let $m, n \geqslant 2$. We apply Lemmas 4 and 5 with $\mu := 2m$ and $\nu := 2n$ to obtain

$$\begin{aligned}
&\frac{1}{\pi^2} \sum_{k=m+1}^{2m} \sum_{\ell=n+1}^{2n} \frac{k\ell C_{k\ell}}{(k-m)(\ell-n)} \\
&\quad \leqslant \left\| \sum_{k=m+1}^{2m} \sum_{\ell=n+1}^{2n} k\ell c_{k\ell} e^{i(kx+\ell y)} \right\| + \left\| \sum_{k=m+1}^{2m} \sum_{\ell=n+1}^{2n} (-k)\ell c_{-k,\ell} e^{i(-kx+\ell y)} \right\| \\
&\qquad + \left\| \sum_{k=m+1}^{2m} \sum_{\ell=n+1}^{2n} k(-\ell) c_{k,-\ell} e^{i(kx-\ell y)} \right\| + \left\| \sum_{k=m+1}^{2m} \sum_{\ell=n+1}^{2n} (-k)(-\ell) c_{-k,-\ell} e^{i(-kx-\ell y)} \right\| \\
&\quad \leqslant 4(2m)(2n) \left\| s_{2m,2n}(f) - s_{m,2n}(f) - s_{2m,n}(f) + s_{mn}(f) \right\|.
\end{aligned}$$

Next, we repeat the above estimates $[m/2]-1$ in place of m in the lower limit of the summation, and with m in place of $2m$ in the upper limit of the summation with respect to k. As a result, we obtain

$$\frac{1}{\pi^2}\sum_{k=[m/2]}^{m}\sum_{\ell=n+1}^{2n}\frac{k\ell C_{k\ell}}{(m-k+1)(\ell-n)}\leqslant 4m(2n)\big\|s_{m,2n}(f)-s_{[m/2]-1,2n}(f)-s_{mn}(f)+s_{[m/2]-1,n}(f)\big\|.$$

The symmetric counterpart of this inequality reads as follows:

$$\frac{1}{\pi^2}\sum_{k=m+1}^{2m}\sum_{\ell=[n/2]}^{n}\frac{k\ell C_{k\ell}}{(k-m)(n-\ell+1)}\leqslant 4(2m)n\big\|s_{2m,n}(f)-s_{mn}(f)-s_{2m,[n/2]-1}(f)+s_{m,[n/2]-1}(f)\big\|.$$

Finally, in a similar manner as above, we obtain

$$\frac{1}{\pi^2}\sum_{k=[m/2]}^{m}\sum_{\ell=[n/2]}^{n}\frac{k\ell C_{k\ell}}{(m-k+1)(n-\ell+1)}\leqslant 4mn\big\|s_{mn}(f)-s_{[m/2]-1,n}(f)-s_{m,[n/2]-1}(f)+s_{[m/2]-1,[n/2]-1}(f)\big\|.$$

Adding up the last four inequalities yields

$$\begin{aligned}&\frac{1}{\pi^2}\sum_{k=[m/2]}^{2m}\sum_{\ell=[n/2]}^{2n}\frac{C_{k\ell}}{(|k-m|+1)(|\ell-n|+1)}\\&\quad\leqslant 16\max_{[m/2]-1\leqslant\mu_1<\mu_2\leqslant 2m}\ \max_{[n/2]-1\leqslant\nu_1<\nu_2\leqslant 2n}\big\|s_{\mu_2,\nu_2}(f)-s_{\mu_1,\nu_2}(f)-s_{\mu_2,\nu_1}(f)+s_{\mu_1,\nu_1}(f)\big\|.\end{aligned}\tag{4.2}$$

By assumption (2.2) (recall notation (4.1)), inequality (4.2) implies (2.3) to be proved. □

Proof of Corollary 2. Let $m,n\geqslant 2$. Denote by $S(m,n)$ the double sum on the left-hand side of (2.3), and consider the arithmetic mean

$$\begin{aligned}A(m,n)&:=\frac{1}{4mn}\sum_{j_1=m}^{3m-1}\sum_{j_2=n}^{3n-1}S(j_1,j_2)\\&=\frac{1}{4mn}\sum_{j_1=m}^{3m-1}\sum_{j_2=n}^{3n-1}\sum_{k=[j_1/2]+1}^{2j_1}\sum_{\ell=[j_2/2]+1}^{2j_2}\frac{C_{k\ell}}{(|k-j_1|+1)(|\ell-j_2|+1)},\end{aligned}\tag{4.3}$$

where $C_{k\ell}$ is defined in (4.1). By assumption (2.3),

$$S(j_1,j_2)\to 0\quad\text{as } j_1,j_2\to\infty.$$

Consequently, we also have

$$A(m,n)\to 0\quad\text{as } m,n\to\infty.\tag{4.4}$$

Now, interchanging the order of summations in (4.3) relating to j_1 and k, as well as relating to j_2 and ℓ, gives the following inequality:

$$\begin{aligned}A(m,n)&=\frac{1}{4mn}\sum_{k=[m/2]}^{6m-2}\sum_{\ell=[n/2]}^{6n-2}C_{k\ell}\sum_{j_1=\max\{[k/2],m\}}^{\min\{2k,6m-2\}}\ \sum_{j_2=\max\{[\ell/2],n\}}^{\min\{2\ell,6n-2\}}\frac{1}{(|k-j_1|+1)(|\ell-j_2|+1)}\\&\geqslant\frac{1}{4mn}\sum_{k=m}^{2m}\sum_{\ell=n}^{2n}C_{k\ell}\sum_{j_1=k}^{k+m}\sum_{j_2=\ell}^{\ell+n}\frac{1}{(j_1-k+1)(j_2-\ell+1)}\\&\geqslant\frac{(\ln(m+2))(\ln(n+2))}{4mn}\sum_{k=m}^{2n}\sum_{\ell=n}^{2n}C_{k\ell}.\end{aligned}\tag{4.5}$$

Combining (4.4) and (4.5) (recall notation (4.1)), yields (2.4) to be proved. □

Acknowledgment

The author thanks the referee for his/her careful reading of the paper and his/her suggestion to include Remark 3 in Section 2.

References

[1] A.S. Belov, On conditions for the mean convergence (boundedness) of partial sums of trigonometric series, in: Metric Theory of Functions and Related Problems in Analysis, Izd. AFT, Moscow, 1999, pp. 1–17 (in Russian); English translation in J. Math. Sci. 155 (1) (2008) 5–17.
[2] A.S. Belov, Remarks on the convergence (boundedness) in the mean of partial sums of a trigonometric series, Mat. Zametki 71 (6) (2002) 807–817 (in Russian); English translation in Math. Notes 71 (5–6) (2002) 739–748.
[3] P.L. Duren, Theory of H^p Spaces, Academic Press, New York, London, 1970.
[4] A. Zygmund, Trigonometric Series, Cambridge Univ. Press, 1959.